

Comments on Lagrange Partial Differential Equation

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Abstract

The relations between solutions of the three types of totally linear partial differential equations of first order are presented. The approach is based on factorization of a non-homogeneous first order differential operator to products consisting of a scalar function, a homogeneous first order differential operator and the reciprocal of the scalar function. The factorization procedure is utilized to show that all totally linear differential equations of first order can be transformed to each other, and in particular to a homogeneous one.

1 Introduction

The method for solution of Lagrange partial differential equation is well known, and is found almost in every text book on partial differential equations[2, 3, 4]. Our goal in this article is to show how the factorization of a non-homogeneous first order differential operator leads quite naturally to simple relations between the solutions of three related types of Lagrange equations.

Let E be an open subset of \mathbb{R}^n , L be a continuous vector field in E , and denote by $C^k(E)$ the set of real valued functions which are continuously differentiable of order k on E . The continuous vector field L may be viewed as a

differential operator [1] from $C^1(E)$ to $C^0(E)$. For each real valued continuous function $q : E \rightarrow \mathbb{R}$ there corresponds an operator $L+q : C^1(E) \rightarrow C^0(E)$ defined by $(L+q)\psi = L\psi + q\psi, \forall \psi \in C^1(E)$, where $L\psi$ is the Lie derivative of the function ψ with respect to the field L , and $q\psi$ is the usual product of two functions. Our goal in this work is to study the relations between the solutions of the partial differential equations

$$(i) L\phi = 0 \quad (ii) (L+q)\psi = 0, \quad (iii) (L+q)\chi = b,$$

where b is a continuous function on E . The approach followed here hinges on factorization of the first order non-homogeneous differential operator $(L+q)$ to a product of a scalar function, a homogeneous differential operator, and the reciprocal scalar function.

2 Factorization of a First Order Non-Homogeneous Operator

Let $\eta \in C^1(E)$ be a non-zero solution of the differential equation

$$(L+q)\eta = 0. \quad (1)$$

Equivalently, η is any element in the kernel of the linear operator $L+q$ that is different from zero. As a first step we assume that η has no zeros in E , and hence η^{-1} exists and of class $C^1(E)$. The general case in which η vanishes on a subset $\delta \subset E$ will be considered in section 4. We start by proving a useful operator equality on which hinges the method of reducing one type of Lagrange equations to another.

Theorem 1 *In $C^1(E)$ the following operator equality holds*

$$\eta L\eta^{-1} = L + q. \quad (2)$$

Proof: for every $\psi \in C^1(E)$.

$$(\eta L\eta^{-1})\psi = \eta(\eta^{-1}L - \eta^{-2}(L\eta))\psi = (L - \eta^{-1}(L\eta))\psi = (L+q)\psi.$$

We have used equation (1) to make the last step.

Corollary 1 *Equality (3) is equivalent to*

$$L = \eta^{-1}(L + q)\eta \quad (3)$$

which shows that all operators of the form $(L + q)$ which are based on the same field L may be transformed to L , and accordingly to each other.

Corollary 2 *The equality (2) shows that the left hand-side must not be dependent on the particular solution η of equation (1), since its right hand-side is not. Therefore if ξ is another solution of (1), then by equation (2) and a similar equation written for the solution ξ , we have $\xi^{-1}\eta L\eta^{-1}\xi = L$. This yields $L(\xi/\eta) = 0$.*

Corollary 3 *From (3) we deduce that*

$$L^k = \eta^{-1}(L + q)^k\eta \quad \text{and} \quad (L + q)^k = \eta L^k \eta^{-1} \quad (4)$$

where k is a non-negative integer. If L is invertible then the latter relation holds for all integers. We assume in relation (4) that L is a C^{k-1} vector field, q is a C^{k-1} function and η is a C^k function.

Corollary 4 *If (1) holds then it is easily checked that $(L + kq)\eta^k = 0$, and hence*

$$L = \eta^{-k}(L + kq)\eta^k \quad (5)$$

In general, and for any real number α , we have

$$L = |\eta|^{-\alpha} (L + \alpha q) |\eta|^\alpha \quad (6)$$

Corollary 5 *If Q is a real valued continuous function on E then*

$$\eta^{-1}(L + q + Q)\eta = L + Q. \quad (7)$$

Corollary 6 Take $Q = -\lambda$ ($\lambda \in R$) in corollary (4) to obtain

$$(L - \lambda)\psi_\lambda = 0 \Leftrightarrow (L + q - \lambda)(\eta\psi_\lambda) = 0 \quad (8)$$

The last relation states that: if ψ_λ is an eigenfunction of the operator L belonging to the eigenvalue λ then $\eta\psi_\lambda$ is an eigenfunction of the operator $L + q$ belonging to the same eigenvalue λ .

Example 1 Take $L + q = \frac{d}{dx} + 2x : C^1(R) \rightarrow C^0(R)$. Since $\eta = e^{-x^2}$ is a solution of (1), we have

$$\frac{d}{dx} + 2x = e^{-x^2} \frac{d}{dx} e^{x^2}$$

It is obvious that every complex number λ is an eigenvalue of the operator $\frac{d}{dx}$ to which an eigenfunction $\psi_\lambda = e^{\lambda x}$ belongs. In accordance with the last corollary, it is easily checked that λ is an eigenvalue of the operator $\frac{d}{dx} + 2x$ to which the function $\eta\psi_\lambda = e^{-x^2 + \lambda x}$ belongs.

3 On Totally Linear Partial Differential Equations

Let (x_1, \dots, x_n) be a global system of coordinates on the region E , in which L is expressed as

$$L = \sum_{k=1}^n a_k(x_1, \dots, x_n) \partial/\partial x_k, \quad (9)$$

where the components a_k are of class C^0 on E . We shall describe the partial differential equation

$$(L + q)\chi = b \quad (10)$$

where $q(x)$ and $b(x)$ are continuous functions on E , as totally linear. The totally linear equation

$$(L + q)\psi = 0, \quad (11)$$

will be referred to as the non-homogeneous reduced equation corresponding to (10), or simply, as the non-homogeneous equation. Equation (10) is a special type of Lagrange equation. The method of solution of Lagrange

equation (10), and consequently equations (10) and (11) is well known [2]. However we aim here to utilize equality (2) to reduce the non-homogeneous equation (11) to the homogeneous equation

$$L\phi = 0, \quad (12)$$

and express its general solution in terms of a particular solution and the general solution of (12). Alternatively, to express the general solution of (11) in terms of n particular solutions. The results we have just pointed to are expressed in the following facts in which we assume that η is a solution of (11) on E and that it has no zeros on E .

F1. A function ϕ_0 is a solution on E of $L\phi = 0$ if and only if $\psi_0 = \eta\phi_0$ is a solution of $(L + q)\psi = 0$ on E .

The proof is a direct consequence of corollary 1 in the previous section.

F2. Let $Q : E \rightarrow \mathbb{R}$ be continuous. By corollary 5 in the previous section, a function ψ_0 is a solution of (11) on E iff $\Psi_0 = \eta\psi_0$ is a solution of $(L + q + Q)\Psi = 0$ on E . In a more familiar language to the subject of differential equation, the transformation $\Psi = \eta\psi$ reduces the last equation to (11).

F3. If η and ξ are solutions of (11) then ξ/η is a solution of (12). Indeed, from equation (11) which is satisfied by η and ξ we get $\eta L\xi = \xi L\eta$, and hence

$$L(\xi/\eta) = \eta^{-2}(\eta L\xi - \xi L\eta) = 0.$$

F4. The general solution of the reduced non-homogenous equation (11) is given by

$$\psi = \eta f(\phi_1, \dots, \phi_{n-1}), \quad (13)$$

where

$$\phi_i = \phi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n-1) \quad (14)$$

are $(n-1)$ functionally independent solutions of (12).

Proof. According to the standard method in solving Lagrange equation [2], the general integral of the homogeneous equation (12) is given by

$$\phi = f(\phi_1, \dots, \phi_{n-1}), \quad (15)$$

where f is an arbitrary C^1 function in its arguments. Now if ψ is a solution of (11) then by F3 ψ/η is a solution of (12), and hence it must be of the form (15). It follows that the general solution of (11) is given by (13).

F5. If

$$\eta(x_1, \dots, x_n), \eta_j(x_1, \dots, x_n) \quad (j = 1, \dots, n-1) \quad (16)$$

are functionally independent solutions of (11) then the ratios

$$\phi_j = \eta_j/\eta \quad (j = 1, \dots, n-1) \quad (17)$$

are solutions of (12). It is easily seen that these ratios are functionally independent. Hence

$$\phi = f(\eta_1/\eta, \dots, \eta_{n-1}/\eta) \quad (18)$$

is the general solution of the homogeneous equation (12), and

$$\psi = \eta f(\eta_1/\eta, \dots, \eta_{n-1}/\eta) \quad (19)$$

is the general solution of the non-homogeneous equation (11). If η_n is a further solution of (11) then by (19)

$$\eta_n/\eta = f_0(\eta_1/\eta, \dots, \eta_{n-1}/\eta), \quad (20)$$

where f_0 is some specified C^1 function.

F6. Let χ_0 be a solution of the totally linear equation (10). If χ is any other solution of (10) then $(L + q)(\chi - \chi_0) = 0$. Hence $\psi = \chi - \chi_0$ is a solution of (11), and consequently must be of the form (13). It follows that the general solution of (10) is given by

$$\chi = \chi_0 + \eta f(\phi_1, \dots, \phi_{n-1}). \quad (21)$$

If a second particular solution χ_1 of (10) is given, then $\eta = \chi_1 - \chi_0$ is a solution of (11), and therefore the general solution of (10) can be written as follows

$$(\chi - \chi_0)/(\chi_1 - \chi_0) = f(\phi_1, \dots, \phi_{n-1}). \quad (22)$$

If χ_2 is a third solution of (10), then by (22), $(\chi_2 - \chi_1)/(\chi_1 - \chi_0)$ is a solution of (12) and has accordingly the form (15). If it is known $(n+1)$ solutions $\chi_0, \chi_1, \dots, \chi_n$ of (10), then

$$\eta = \chi_1 - \chi_0, \eta_1 = \chi_2 - \chi_0, \dots, \eta_{n-1} = \chi_n - \chi_0$$

are solutions of (11), and hence

$$\phi_1 = \frac{\eta_1}{\eta} = \frac{\chi_2 - \chi_0}{\chi_1 - \chi_0}, \dots, \phi_{n-1} = \frac{\eta_{n-1}}{\eta} = \frac{\chi_n - \chi_0}{\chi_1 - \chi_0} \quad (23)$$

are solutions of (12). If these ratios are functionally independent then

$$\frac{\chi - \chi_0}{\chi_1 - \chi_0} = f\left(\frac{\chi_2 - \chi_0}{\chi_1 - \chi_0}, \dots, \frac{\chi_n - \chi_0}{\chi_1 - \chi_0}\right) \quad (24)$$

gives the general solution χ of (10) in terms of $(n+1)$ particular solutions. It is clear of course that these $(n+1)$ solutions of (10) are certainly functionally dependent. If χ_{n+1} is a further solution of (10) then by (24) $(\chi_{n+1} - \chi_0)/(\chi_1 - \chi_0)$ is some function of the solutions (23) of equation (12).

4 The General Case

We proceed here to study the general case in which the solution η of equation (1) is defined only on an open subset $E_\eta \subset E$, and vanishes on a closed subset $\delta \subset E$. This latter assumption embodies most interesting cases which are encountered in practical examples, such as η vanishing at a finite or a countable set, or outside an open subset of its domain of definition. In the present case the operator $\eta L \eta^{-1}$ exists only on $E_\eta - \delta$. Hence the equality (2) must be replaced by the inclusion relation $(\eta L \eta^{-1} \subset L + q)$; it is an equality only on $E_\eta - \delta$.

Some minor modification has to be made so that the corollaries of theorem 1 in section 2 and the facts in section 3 conform to the new assumptions. As an example the fact F1 in section 3 must be modified as follows:

F1. Let η be a solution of (1) on E_η which vanishes on a closed subset $\delta \subset E_\eta$.

- (i) ϕ is a solution of (12) on $E_\phi \Rightarrow \eta\phi$ is a solution of (11) on $E_\phi \cap E_\eta$.
- (ii) ψ is a solution of (11) on $E_\psi \Rightarrow \psi/\eta$ is a solution of (12) on $E_\psi \cap E_\eta - \delta$.

Proof: (i) on $E_\phi \cap E_\eta$, where η is defined, we have

$$(L + q)(\eta\phi) = \phi(L + q)\eta + \eta L\phi.$$

If $L\phi = 0$ on E_ϕ then $(L + q)(\eta\phi) = 0$ on $E_\eta \cap E_\phi$.

(ii) On $E_\phi \cap E_\eta - \delta$, where ψ/η is defined and is of class C^1 , we have $L(\psi/\eta) = 0$, as it was shown in F3.

It must be noted that the last fact determines the smallest domains on which ϕ and ψ are defined. These domains may be extended by continuity to larger domains.

The remaining facts and corollaries can be modified in a similar way.

Example. Consider the operator $L = x\partial/\partial x + y\partial/\partial y + z\partial/\partial z$ which is continuous on \mathfrak{R}^3 . The general solution of the equation $L\phi = 0$ is $\phi = f(y/x, z/x)$, where f is an arbitrary C^1 function. The equation $(L + 3/2)\psi = 0$ admits the particular solution $\eta = x^{-3/2}$ on $E_\eta = \{(x, y, z) \in \mathfrak{R}^3 : x > 0\}$. The function $\phi = x/y$ is a solution of $L\phi = 0$ on $E_\phi = \{(x, y, z) \in \mathfrak{R}^3 : y \neq 0\}$, and the function $\psi = \eta\phi = x^{-1/2}y^{-1}$ is a solution of $(L + 3/2)\psi = 0$ on $\{(x, y, z) \in \mathfrak{R}^3 : y \neq 0, x > 0\} = E_\eta \cap E_\phi$. It is clear that we could have adopted $\eta' = |x|^{-3/2}$, and $\phi' = |x|/y$ (since L is linear), to obtain a solution $\psi' = |x|^{-1/2}y^{-1}$ defined on $\mathfrak{R}^3 - \{x = 0, y = 0\} = E_{\eta'} \cap E_{\phi'}$. If we replace in the given equation $3/2$ by 1 , then $\eta = 1/x$, and hence $\eta\phi = 1/y$ is a solution of $(L + 1)\psi = 0$ on E_ϕ .

Consider now the totally linear equation $(L + 1)\chi = 3x^2$. Four particular solutions of this equation are $\chi_0 = x^2$, $\chi_1 = x^2 + x^{-1}$, $\chi_2 = x^2 + y^{-1}$, $\chi_3 = x^2 + z^{-1}$. It can be checked that the formula (24) yields, in accordance with (21), the general solution $\chi = x^2 + x^{-1}f(y/x, z/x)$.

5 An Algebraic Comment

Let A denote the commutative real algebra formed by the set of all functions defined in E . The set of solutions of the homogeneous equation $L\phi = 0$, namely $\text{Ker } L$, is clearly a subalgebra of A . The set of solutions of non-homogeneous equation (11) is a coset of this subalgebra determined by a particular solution of (11). If η is a particular solution of (11) defined every where in E , then $\text{Ker}(L + q) = \eta \text{Ker } L$. It is clear that $\text{Ker}(L + q)$ is a sub-vector space of A . The set of solutions of equation (10) is a coset of the sub-vector space $\text{Ker}(L + q)$ determined by any particular solution of (10).

References

- [1] Abraham R and Marsden J, Reading, Mass., Benjamin/Cummings Pub. Co. (1978).
- [2] Piaggio H, An Elementary Treatise on Differential Equations and Their Applications. G. Bell and Sons LTD, London (1969).

- [3] Ayres F. Differential Equations. Schaum's Outline Series. McGraw Hill Book Company, New York (1952).
- [4] Carrier G F, Partial Differential Equations. Academic Press, New York (1979).